$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Assumption:

- One dimension heat flow r-direction.
- Steady state.
- No heat generation.

For the control volume in the figure energy conservation required that

$$q_r = q_{r+dr}$$

The appropriate form of Fouries low is

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

The general heat equation reduced to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)=0$$

$$\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)=0$$

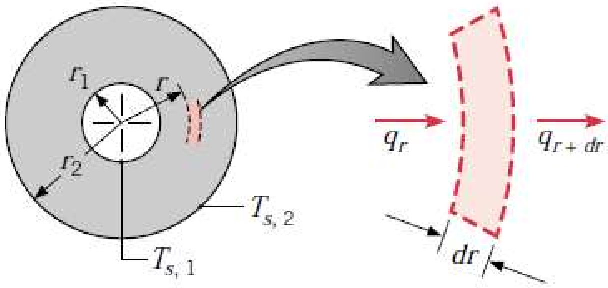


Figure 2.6: conduction in spherical shell

First integration:

$$r^2\frac{\partial T}{\partial r}=C_1$$

$$\frac{\partial T}{\partial r}=\frac{C_1}{r^2}$$

Second integration:

$$\boldsymbol{T(r)} = -\frac{C_1}{r}+C_2 \qquad \text{..........a}$$

B.C: at $r = r_1$, $T = T_{s,1}$

at $r = r_2$, $T = T_{s,2}$

apply B.C in equation a

$$T_{s,1} = -\frac{C_1}{r_1}+C_2 \qquad \text{..........b}$$

$T_{s,2} = -\frac{C_1}{r_2} + C_2$ c

Solve equation b and c we get:

$$C_1 = \frac{(T_{s,1}-T_{s,2})}{\frac{1}{r_2}-\frac{1}{r_1}}$$

$$C_2 = T_{s,1} - \frac{1}{r_1}\frac{(T_{s,1}-T_{s,2})}{\frac{1}{r_1}-\frac{1}{r_2}}$$

Sub. C_1& C_2 in equation a, for temperature distribution:

$$\frac{T(r)-T_{s,1}}{T_{s,2}-T_{s,1}} = \frac{r_2}{r}\left(\frac{r-r_1}{r_2-r_1}\right)$$

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r^2}, A = 4\pi r^2$$

$$q_r = -4\pi k r^2 \frac{(T_{s,1}-T_{s,2})}{r^2(\frac{1}{r_2}-\frac{1}{r_1})}$$

$$q_r = 4\pi k \frac{(T_{s,1}-T_{s,2})}{(\frac{1}{r_1}-\frac{1}{r_2})}$$

$$R_{t,cond} = \frac{\frac{1}{r_1}-\frac{1}{r_2}}{4\pi k}$$

2.3 Conduction with thermal energy generation:

2.3.1 Plane wall with heat generation:

To investigate the effect of uniform internal heat generation in one dimension temperature distribution within the plane wall. Figure 2.7 shows the plane wall with internal uniform heat generation, $\dot{q}$.

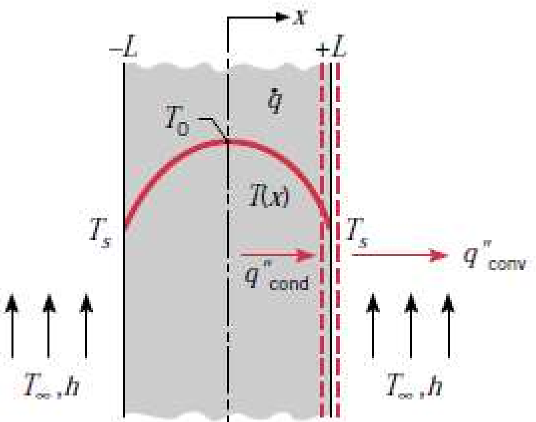


Figure 2.7 Conduction in a plane wall with uniform heat generation with Symmetrical boundary conditions.

The general conduction equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

For study, one dimension:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0$$

First integration gives:

$$\frac{\partial T}{\partial x} = -\frac{\dot{q}x}{k} + C_1$$

Second integration gives:

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2 \quad \text{................a}$$

B.C: at $x = L, T = T_s$: at $x = -L, T = T_s$

Apply B.C to equation a:

$$T_s = -\frac{\dot{q}}{2k}L^2 + C_1 L + C_2 \quad \text{.........b}$$

$$T_s = -\frac{\dot{q}}{2k}L^2 - C_1 L + C_2 \quad \text{...........c}$$

So:

$C_2 = T_s + \frac{\dot{q}}{2k}L^2$ and $C_1 = 0$

Sub. $C_1 \& C_2$ in equation a gives:

$$T(x) - T_s = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) \quad \text{..........d}$$

This is a parabolic temperature distribution that is symmetry about x=0

Thus at x=0, $\frac{\partial T}{\partial x} = 0$

And the maximum temperature axis at x=0

$T_{max} - T_s = \frac{\dot{q}}{2k} L^2$ ....e

From equation d

$$\frac{\dot{q}}{2k} L^2 = \frac{T(x) - T_s}{\left(1 - \frac{x^2}{L^2}\right)}$$

Sub. In equation e we get:

$$\frac{T(x) - T_s}{T_{max} - T_s} = 1 - \frac{x^2}{L^2}$$

in case the temperature distribution is asymmetrical as shown in Figure 2.8 the temperature distribution is as below:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

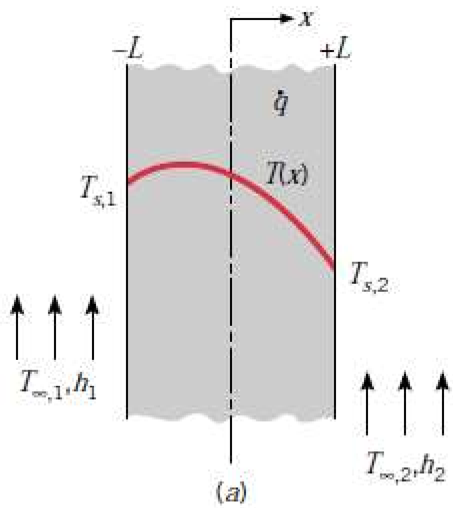


Figure 2.8: Conduction in a plane wall with uniform heat generation with Asymmetrical boundary conditions.

2.3.2 Cylinder with heat generation:

Heat generation may occur in a variety of radial geometries. Consider the long, solid cylinder of Figure 2.9, which could represent a current-carrying wire or a fuel element in a nuclear reactor. For steady-state conditions the rate at which heat is generated within the cylinder must equal the rate at which heat is convected from the surface of the cylinder to a moving fluid. This condition allows the surface temperature to be maintained at a fixed value of T_s.

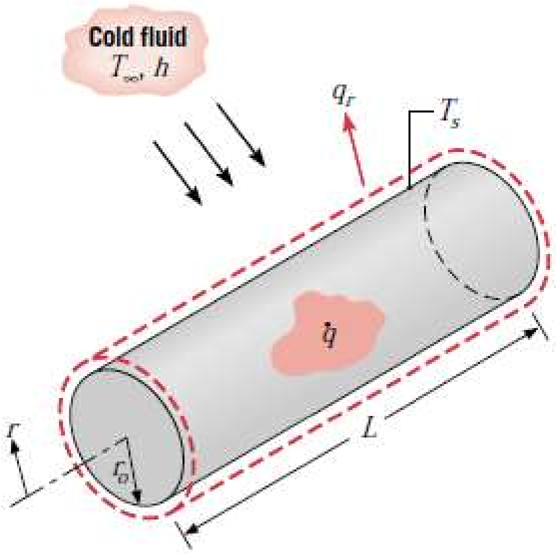


FIGURE 2.9 Conduction in a solid cylinder with uniform heat generation.

To determine the temperature distribution in the cylinder, we begin with the appropriate form of the heat equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

For constant thermal conductivity k, the heat conduction Equation reduces to:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Separating variables and assuming uniform generation, this expression may be integrated to obtain

$$r\frac{dT}{dr} = -\frac{\dot{q}}{2k}r^2 + C_1$$

Repeating the procedure, the general solution for the temperature distribution becomes

$$T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 lnr + C_2 \qquad ....a$$

To obtain the constants of integration C_1 and C_2, we apply the boundary conditions

$$\frac{dT}{dr} = 0 \; at \; r = 0 \, , T = T_s \; at \; r = r_o$$

Therefore:

$$C_1 = 0$$

$$C_2 = T_s + \frac{\dot{q}}{4k}r_o^2$$

The temperature distribution is therefore

$$T(r) - T_s = \frac{\dot{q}r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) \qquad ....b$$

$$T = T_{max} \; at \; r = 0$$

Sub in equation

$$T_{max} - T_s = \frac{\dot{q}r_o^2}{4k} \qquad ....c$$

Divided equation (b) to equation (c) we obtained

$$\frac{T(r)-T_s}{T_{max}-T_s} = 1 - \left(\frac{r}{r_o}\right)^2$$

From equation (b)

$$\frac{dT}{dr} = -\frac{\dot{q}}{2k}r$$

$$q(r) = -k2\pi rl\frac{dT}{dr}$$

$q(r) = -k2\pi rL(-\frac{\dot{q}}{2k}r)$ (heat conduction in the radius direction)

$$q(r) = \pi L\dot{q}r^2 \qquad ....d$$

$$q(r_o) = \pi L\dot{q}r_o^2$$

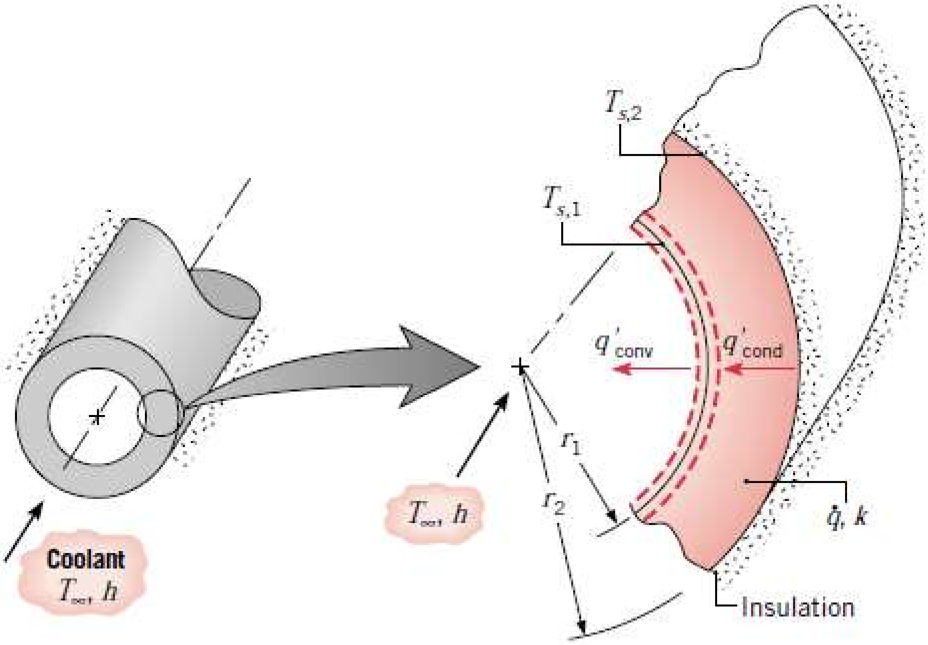
To relate the surface temperature, T_s, to the temperature of the cold fluid, T , either a surface energy balance or an overall energy balance may be used.

$$q_{cond} = q_{conv}$$

$$\pi L\dot{q}r_o^2 = h(2\pi r_o L)(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}r_o}{2h}$$

H.W a similar analysis applies to a hollow cylinder as shown in the figure below:



Boundary condition: at $r = r_2$ $T(r) = T_{s,2}$: $r = r_2$ $\frac{dT}{dr} = 0$

Example:

A current of 200 A is passed through a stainless-steel wire [k = 19 W/m · ◦C] 3 mm in diameter. The resistivity of the steel may be taken as 70 μΩ · cm, and the length of the wire is 1 m. The wire is submerged in a liquid at 110 °C and experiences a convection heat-transfer coefficient of 4kW/m^2 · ◦C. Calculate the center temperature of the wire.

Solution:

All the power generated in the wire must be dissipated by convection to the liquid:

$$P = I^2R = q = hA(T_s - T_\infty) \quad ....a$$

The resistance of the wire is calculated from

$R = \rho \frac{L}{A_c} = \frac{(70x10^{-6})(100)}{\pi(0.15)^2} = 0.099\Omega$

where ρ is the resistivity of the wire. The surface area of the wire is πdL, so from Equation (a)

$(200)^2(0.099) = 4000\pi(3x10^{-3})(1)(T_s - 110)$

T_s=215˚C.

The heat generated per unit volume $\dot{q}$ is calculated from

$P = \dot{q}V = \dot{q}\pi r_o^2 L$

So that,

$\dot{q} = \frac{3960}{\pi(1.5x10^{-3})^2(1)} = 560.2\ MW/m^3$

Finally, the center temperature of the wire is calculated from Equation below:

$T_{max} - T_s = \frac{\dot{q}r_o^2}{4k}$

$T_{max} = \frac{(5.605x10^8)(1.5x10^{-3})^2}{(4)(19)} + 215 = 231.6\ ℃$

2.3.3 sphere with heat generation

The general heat conduction equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

For one dimension (r direction only) study state the general equation has been reduced to:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$$

Boundary condition: at $r = 0$ $\frac{dT}{dr} = 0$ & $at\ r = r_o\ T = T_s$

$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\dot{q}}{k}r^2$

First integration gives:

$r^2 \frac{dT}{dr} = -\frac{\dot{q}}{3k} r^3 + C_1$

$\frac{dT}{dr} = -\frac{\dot{q}}{3k} r + \frac{C_1}{r^2}$

Second integration gives:

$T(r) = -\frac{\dot{q} r^2}{6k} - \frac{C_1}{r} + C_2$

H.W find the temperature distribution and heat transfer.

Q1: Air flows at 120◦C in a thin-wall stainless-steel tube with h=65 W/m^2 · ◦C. The inside diameter of the tube is 2.5 cm and the wall thickness is 0.4 mm. k =18 W/m · ◦C for the steel. The tube is exposed to an environment with h=6.5 W/m^2 · ◦C and T_∞ =15◦C. Calculate the overall heat-transfer coefficient and the heat loss per meter of length.

Solution:

For $L = 1$ m $\quad \frac{1}{h_i A_i} = \frac{1}{(65)\pi(0.025)} = 0.1959$

$$\frac{\ln(r_0/r_i)}{2\pi k} = \frac{\ln(2.58/2.5)}{2\pi(18)} = 2.79 \times 10^{-4}$$

$$\frac{1}{h_0 A_0} = \frac{1}{(6.5)\pi(0.0258)} = 1.898$$

$$UA = \frac{1}{\sum R} = 2.094$$

$$\frac{q}{L} = (2.094)(120 - 15) = 219.9 \ \frac{\text{W}}{\text{m}}$$

Q2: An insulating glass window is constructed of two 5-mm glass plates separated by an air layer having a thickness of 4 mm. The air layer may be considered stagnant so that pure conduction is involved. The convection coefficients for the inner and outer surfaces are 12 and 50 W/m^2 . ◦C, respectively. Calculate the overall heat-transfer coefficient for this arrangement. Assume thermal conductivity of glass equal to 0.78 W/m . ◦C, and for air 0.026 W/m^2 . ◦C

Solution:

$$A = 1\ \text{m}^2 \qquad R_{\text{glass}} = \frac{\Delta x}{k} = \frac{0.005}{0.78} = 6.41\times10^{-3}$$

$$R_{\text{air}} = \frac{\Delta x}{k} = \frac{0.004}{0.026} = 0.1538$$

$$R_{\text{conv}_1} = \frac{1}{h} = \frac{1}{12} = 0.0833$$

$$R_{\text{conv}_2} = \frac{1}{50} = 0.02$$

$$U = \frac{1}{(2)(6.41\times10^{-3}) + 0.1538 + 0.0833 + 0.02} = \frac{1}{0.2699} = 3.705\ \frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}$$

Q3: A wall consists of a 1-mm layer of copper, a 4-mm layer of 1 percent carbon steel, a 1-cm layer of asbestos sheet, and 10 cm of fiberglass blanket. Calculate the overall heat-transfer coefficient for this arrangement. If the two outside surfaces are at 10 and 150◦C, calculate each of the interface temperatures.

Solution:

$$R_{Cu} = \frac{0.001}{386} = 2.59\times10^{-6} \qquad \Delta T_{Cu} = (52)(2.59\times10^{-6}) = 1.35\times10^{-4}\,{}^\circ\text{C}$$

$$R_{St} = \frac{0.004}{43} = 9.3\times10^{-5} \qquad \Delta T_{St} = (52)(9.3\times10^{-5}) = 4.84\times10^{-3}\,{}^\circ\text{C}$$

$$R_{As} = \frac{0.01}{0.166} = 0.0602 \qquad \Delta T_{As} = (52)(0.0602) = 3.13{}^\circ\text{C}$$

$$R_F = \frac{0.1}{0.038} = 2.632 \qquad \Delta T_F = (52)(2.632) = 136.9{}^\circ\text{C}$$

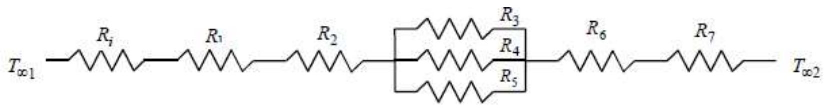
$$\sum R = 2.692 \qquad U = \frac{1}{R} = 0.371\ \frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}$$

$$q = U\Delta T = (0.371)(150-10) = 52\ \frac{\text{W}}{\text{m}^2}$$

Inside of copper = 150°C

3–52 A 4-m-high and 6-m-wide wall consists of a long 18-cm 30-cm cross section of horizontal bricks (k _ 0.72 W/m · °C) separated by 3-cm-thick plaster layers (k = 0.22 W/m · °C). There are also 2-cm-thick plaster layers on each side of the wall, and a 2-cm-thick rigid foam (k = 0.026 W/m · °C) on the inner side of the wall. The indoor and the outdoor temperatures are 22°C and - 4°C, and the convection heat transfer coefficients on the inner and the outer sides are h_1 = 10 W/m2 · °C and h_2 = 20 W/m2 · °C, respectively. Assuming one-dimensional heat transfer and disregarding radiation, determine the rate of heat transfer through the wall.

Solution:



$$R_i = R_{conv,1} = \frac{1}{h_1 A} = \frac{1}{(10\ \text{W/m}^2.°\text{C})(0.33\times 1\ \text{m}^2)} = 0.303\ °\text{C/W}$$

$$R_1 = R_{foam} = \frac{L}{kA} = \frac{0.02\ \text{m}}{(0.026\ \text{W/m}.°\text{C})(0.33\times 1\ \text{m}^2)} = 2.33\ °\text{C/W}$$

$$R_2 = R_6 = R_{\substack{plaster\\ side}} = \frac{L}{kA} = \frac{0.02\ \text{m}}{(0.22\ \text{W/m}.°\text{C})(0.30\times 1\ \text{m}^2)} = 0.303\ °\text{C/W}$$

$$R_3 = R_5 = R_{\substack{plaster\\ center}} = \frac{L}{h_o A} = \frac{0.18\ \text{m}}{(0.22\ \text{W/m}.°\text{C})(0.015\times 1\ \text{m}^2)} = 54.55\ °\text{C/W}$$

$$R_4 = R_{brick} = \frac{L}{kA} = \frac{0.18\ \text{m}}{(0.72\ \text{W/m}.°\text{C})(0.30\times 1\ \text{m}^2)} = 0.833\ °\text{C/W}$$

$$R_o = R_{conv,2} = \frac{1}{h_2 A} = \frac{1}{(20\ \text{W/m}.°\text{C})(0.33\times 1\ \text{m}^2)} = 0.152\ °\text{C/W}$$

$$\frac{1}{R_{mid}} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} = \frac{1}{54.55} + \frac{1}{0.833} + \frac{1}{54.55} \longrightarrow R_{mid} = 0.81\ °\text{C/W}$$

$$R_{total} = R_i + R_1 + 2R_2 + R_{mid} + R_o = 0.303 + 2.33 + 2(0.303) + 0.81 + 0.152$$
$$= 4.201\ °\text{C/W}$$

The steady rate of heat transfer through the wall per $0.33\ \text{m}^2$ is

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} = \frac{[(22-(-4)]°\text{C}}{4.201°\text{C/W}} = 6.19\ \text{W}$$

Then steady rate of heat transfer through the entire wall becomes

$$\dot{Q}_{total} = (6.19\ \text{W})\frac{(4\times 6)\text{m}^2}{0.33\ \text{m}^2} = \mathbf{450\ W}$$

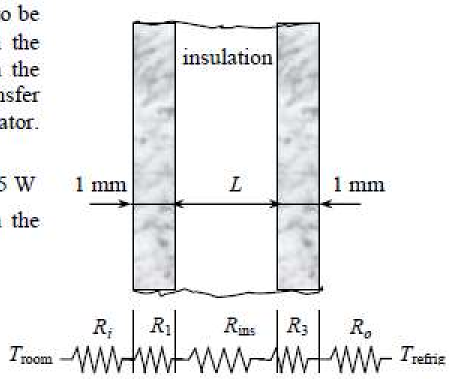
3–35 The wall of a refrigerator is constructed of fiberglass insulation (k = 0.035 W/m · °C) sandwiched between two layers of 1-mm-thick sheet metal (k = 15.1 W/m · °C). The refrigerated space is maintained at 3°C, and the average heat transfer coefficients at the inner and outer surfaces of the wall are 4 W/m^2 · °C and 9 W/m^2 · °C, respectively. The kitchen temperature averages 25°C. It is observed that condensation occurs on the outer surfaces of the refrigerator when the temperature of the outer surface drops to 20°C. Determine the minimum thickness of fiberglass insulation that needs to be used in the wall in order to avoid condensation on the outer surfaces.

Solution:

Analysis The minimum thickness of insulation can be determined by assuming the outer surface temperature of the refrigerator to be 10°C. In steady operation, the rate of heat transfer through the refrigerator wall is constant, and thus heat transfer between the room and the refrigerated space is equal to the heat transfer between the room and the outer surface of the refrigerator. Considering a unit surface area,

$$\dot{Q} = h_o A(T_{room} - T_{s,out}) = (9\ \text{W/m}^2.°\text{C})(1\ \text{m}^2)(25-20)°\text{C} = 45\ \text{W}$$

Using the thermal resistance network, heat transfer between the room and the refrigerated space can be expressed as



$$\dot{Q} = \frac{T_{room} - T_{refrig}}{R_{total}}$$

$$\dot{Q}/A = \frac{T_{room} - T_{refrig}}{\frac{1}{h_o} + 2\left(\frac{L}{k}\right)_{metal} + \left(\frac{L}{k}\right)_{insulation} + \frac{1}{h_i}}$$

Substituting,

$$45\ \text{W/m}^2 = \frac{(25-3)°\text{C}}{\frac{1}{9\ \text{W/m}^2.°\text{C}} + \frac{2\times 0.001\ \text{m}}{15.1\ \text{W/m}^2.°\text{C}} + \frac{L}{0.035\ \text{W/m}^2.°\text{C}} + \frac{1}{4\ \text{W/m}^2.°\text{C}}}$$

Solv ing for L, the minimum thickness of insulation is determined to be

$$L = 0.0045\ \text{m} = \mathbf{0.45\ cm}$$